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COMMON FIXED POINT THEOREM IN B-METRIC – LIKE SPACES

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ABSTRACT

In this paper obtain common fixed point result involving generalized (ψ-φ)- weakly contractive condition in b- metric- like space.

KEYWORDS: *b- Metric Space, Fixed Point, Common Fixed Point, Cauchy Sequence.*

Article History

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INTRODUCTION

The concept of φ- contractive mappings was introduced by Rhoads[16].After word, some researchers introduced a few φ and (ψ-φ)- weakly contractive condition and discussed the existence of fixed an common fixed point for these mapping [see 1,2,4,5,8,10,11,12,13,14,15,18,19].In particular Aghajani et al.[19] presented several common fixed point results of generalized weak contractive mapping in partially order b- metric spaces. Recently Guan et al.[3] introduced idea of b-metric –like space and give some theorems in this metric space. Further some researcher discussed common fixed point theorems in this metric spaces [see 3,5,6,7,9,17]. In this paper obtain common fixed point result involving general (ψ,φ)-weakly contractive to condition in b- metric –like- spaces. We give example to support our results. Obtained results are also generalizations of many theorems.

PRELIMINARIES

Definition [17]

Let X be a non empty set and s≥ 1 be a given real number. A mapping d: X x X→[0,∞) is said to be a b- metric if and only if, for all x,y,z ε X, the following conditions are satisfied

- d(x,y) = 0 if and only if x = y
- d(x,y) = d(y,x)
- d(x,y) = s[d(x,z) + d(z,y)].

The pair (X, d) is called a b-metric space with parameter s ≥ 1.

In general, the class of b-metric space is effectively larger than that of metric space, since a b- metric is a metric with s = 1. We can find several examples of b- metric space which is not metric space (sea [18]).

Definition [9]

Let X be a non empty set and s≥ 1 be a given real number. A mapping d: X x X→[0,∞) is said to be a b- metric – like if and only if, for all x,y,z ε X, the following conditions are satisfied

- d(x,y) = 0 implies if x = y
- d(x,y) = d(y,x)
- d(x,y) = s[d(x,z) + d(z,y)].

The pair (X, d) is called a b-metric- like space with parameter s ≥ 1.

We should note that in a b-metric- like space (X,d) if x,y ε X and d(x,y) = 0 then x = y. But the converse need not be true and d(x,x) may be positive for x ε X.

Example [9]

Let X = [0,∞) and let a mapping d: X x X→[0,∞) be defined by d(x.y) = $(x+y)^2$ for all x,y ε X. Then (X,d) is a b- metric- like space with parameter s ≥2.

Lemma [9]

Let (X,d) be a b- metric –like space with s≥1. We assume that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y respectively,

we have $1/s^2 d(x,y) - 1/s\, d(x,x) - d(y,y) \le \limsup_{n\to\infty} d(x_n,y_n) \le sd(x,x) + s^2\, d(y,y) + s^2 d(x,y)$

In particular, if d(x,y) = 0, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each zεX, we have

$1/sd(x,z) - d(x,x) \le \limsup_{n\to\infty} d(x_n,z) \le sd(x,z) + s\, d(x,x)$

In particular, if d(x,x) = 0, then we have $1/s\, d(x,z) \le \limsup_{n\to\infty} d(x_n,z) \le s\, d(x,z)$.

Lemma [7]

Let (X,d) be a b- metric –like space with s≥1.

Then 1. If d(x,y) = 0, then d(x,x) = d(y,y) = 02.If $\{x_n\}$ is a sequence with that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Then we have $\lim_{n\to\infty} d(x_n,x_n) = \lim_{n\to\infty} d(x_{n+1},x_{n+1}) = 0$

3. If x ≠ y , then d(x,y) > 0.

Theorem [1]

Let(X,d) be a complete b- metric- like- space with parameter s ≥ 1 and let f,g : X→X be self mapping f (X) c g(X) where g(x) is a closed subset of X. If there are function ψ ε Ψ and φ ε Φ such that

$\Psi(s^2[d(fx,fy)]^2) \le \psi(N(x,y)) - \varphi(M(x,y))$,

where $N(x,y) = \max\{ [d(fx,gx)]^2, [d(gx,gy)]^2, [d(fy,gy)]^2, d(fx,gx)d(fx,fy), d(fx,gx)d(gx,gy)$,

and $M(x,y) = \max\{ [d(fy,gy)]^2, [d(fx,gy)]^2, [d(gx,gy)]^2, \frac{[d(fx,gy)]2[1 + [d(gx,gy)]2]}{[1 + [d(fx,gy)]2]}\}$

- ψ: $[0,\infty)\to[0,\infty)$ is a continuous and non-decreasing function with $\psi(t) = 0$ implies $t= 0$.
- φ: $[0,\infty)\to[0,\infty)$ is a continuous and increasing function with $\varphi(t) = 0$ if and only if $t= 0$,

Then f and g have a unique coincidence point in X. Moreover f and g have a unique common fixed point provided that f and g are weakly compatible.

MAIN RESULT

Theorem 3.1

Let(X,d) be a complete b- metric- like- space with parameter $s \geq 1$ and let $f,g : X\to X$ be self mapping $f(X) \subset g(X)$ where g(x) is a closed subset of X. If there are function $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\Psi(s^2[d(fx,fy)]^2) \leq \psi(M(x,y)) - \varphi(M(x,y)),. \quad 3.1$$

Where $M(x,y)=\max\{ [d(fy,gy)]^2, [d(fx,gy)]^2,[d(gx,gy)]^2, \frac{[d(fx,gy)]2[1 + [d(gx,gy)]2]}{[1 + [d(fx,gy)]2]}, \frac{[d(gx,gy)]2[1 + [d(gx,gy)]2]}{[1 + [d(fx,gy)]2]} \}$

- ψ: $[0,\infty)\to[0,\infty)$ is a continuous and non-decreasing function with $\psi(t) = 0$ implies $t= 0$.
- φ: $[0,\infty)\to[0,\infty)$ is a continuous and increasing function with $\varphi(t) = 0$ if and only if $t= 0$,

Then f and g have a unique coincidence point in X. Moreover f and g have a unique common fixed point provided that f and g are weakly compatible.

Proof

Let $x_0 \in X$. As $f(X) \subset g(X)$, there $x_1 \in X$ such that $fx_0 = gx_1$. Now we define the sequence $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in N$. If $y_n= y_{n+1}$ for some $n \in N$, then we have $y_n=y_{n+1}= fx_{n+1}= gx_{n+1}$

And f and g have a coincidence point. Without loss of generality, we assume that $y_n \neq y_{n+1}$ by lemma, we know that $d(y_n,y_{n+1}) > 0$ for all $n \in N$. Applying 3.1 with $x = x_n$and $y = x_{n+1}$, we obtain

$$\Psi(s^2[d(y_n,y_{n+1})]^2) = \Psi(s^2[d(fx_n,fx_{n+1})]^2) \leq \psi(M(x_n,x_{n+1})) - \varphi(M(x_n,x_{n+1}))). \quad 3.2$$

where $M(x_n,x_{n+1}) = \max\{ [d(fx_{n+1},gx_{n+1})]^2, [d(fx_n,gx_{n+1})]^2, [d(gx_n,gx_{n+1})]^2, [d(fx_n,gx_n)]^2 [1+ [d(gx_n,gx_{n+1})]^2]/[1+, [d(fx_n,gx_{n+1})]^2], [d(gx_n,gx_{n+1})]^2 [1+ [d(gx_n,gx_{n+1})]^2]/[1+ [d(fx_n,gx_{n+1})]^2] \}$

$$= \max\{ [[d(y_n,y_{n+1})]^2, [d(y_n,y_n)]^2, [d(y_{n+1},y_n)]^2, [d(y_n,y_{n-1})]^2[1+d(y_{n-1},y_n)]^2]/[1+[d(y_n,y_n)]^2], [d(y_{n-1},y_n)]^2[1+d(y_{n-1},y_n)]^2]/[1+d(y_n,y_{n+1})]^2] \quad 3.3$$

If $d(y_n,y_{n+1}) \geq d(y_n,y_{n-1}) >0$ for some $n \in N$. In view of 3.3, we have

$M(x_n,x_{n+1}) \geq [d(y_n,y_{n+1})]^2$

$= \max\{ [d(y_n,y_{n+1})]^2, [d(y_{n-1},y_n)]^2\}$

$$\Psi(s^2[d(y_n,y_{n+1})]^2) \leq \Psi(s^2[d(y_n,y_{n+1})]^2) \leq \psi(M(x_n,x_{n+1})) - \varphi(M(x_n,x_{n+1}))$$

$$\leq \psi(M(y_n,y_{n+1})) - \varphi(M(y_n,y_{n+1})) \quad 3.4$$

Which implies $\varphi\,[d(y_n,y_{n+1})]^2 = 0$ i.e. $y_n = y_{n+1}$ contradiction.

Hence $d(y_n,y_{n+1}) < d(y_n,y_{n-1})$ and $\{ d(y_n,y_{n+1})\}$ is a non increasing sequence and so there exists $r \geq 0$

$\lim_{n\to\infty} d(y_n,y_{n+1}) = r$.

By 3.3, we have $M(x_n,x_{n+1}) =[d(y_n,y_{n+1})]^2$.

It follows that

$$\Psi(s^2[d(y_n,y_{n+1})]^2) \leq \psi(M(x_n,x_{n+1})) - \varphi(M(x_n,x_{n+1}))$$

$$\leq \psi([d(y_n,y_{n-1})]^2) - \varphi([d(y_n,y_{n-1})]^2).$$

Now suppose that $r > 0$.By taking the lim as $n\to\infty$ in 3.4, we have $\psi(r^2) \leq \psi(r^2) - \varphi(r^2)$a contradiction. This yields that

$$\lim_{n\to\infty} d(y_n,y_{n+1}) = r = 0 \quad 3.5$$

Now we shall prove that $\lim_{n\to\infty}d(y_n,y_m) = 0$. Suppose on the contrary that $\lim_{n\to\infty}d(y_n,y_m) \neq 0$. It follows that there exists $\varepsilon > 0$ for which one can find sequence $\{y_{mk}\}$and $\{y_{nk}\}$ of $\{y_n\}$ where nk is the smallest index for which $n_k>m_k> k$, $\varepsilon \leq d(y_{mk},y_{nk})$, and $d(y_{mk},y_{nk-1}) < \varepsilon$.

In view of the triangle inequality in b- metric- like space, we get

$$\varepsilon^2 \leq [d(y_{mk}, y_{nk})]^2 \leq [sd(y_{mk},y_{nk-1})+ sd(y_{nk-1},y_{nk})]^2$$

$$= s^2 [d(y_{mk},y_{nk-1})]^2+ s^2 [d(y_{nk-1},y_{nk})]^2 +2s^2 d(y_{mk},y_{nk-1})\, d(y_{nk-1},y_{nk})\ldots \quad 3.6$$

$= s^2 \varepsilon^2 + s^2 [d(y_{nk-1},y_{nk})]^2 +2s^2 d(y_{mk},y_{nk-1})\, d(y_{nk-1},y_{nk})$. Using equality 3.5 and taking the upper limit as $k\to\infty$ in the above inequality, we obtain

$$\varepsilon^2 \leq \limsup_{k\to\infty} [d(y_{mk}, y_{nk})]^2 \leq s^2 \varepsilon^2 \quad 3.7$$

As the same arguments, we deduce the following results

$$\varepsilon^2 \leq [d(y_{mk}, y_{nk})]^2 \leq [sd(y_{mk},y_{nk-1})+ sd(y_{nk-1},y_{nk})]^2$$

$$= s^2 [d(y_{mk},y_{nk-1})]^2+ s^2 [d(y_{nk-1},y_{nk})]^2 +2s^2 d(y_{mk},y_{nk-1})\, d(y_{nk-1},y_{nk}) \quad 3.8$$

$$[d(y_{mk}, y_{nk})]^2 \leq [sd(y_{mk},y_{mk-1})+ sd(y_{mk-1},y_{nk})]^2$$

$$= s^2 [d(y_{mk},y_{mk-1})]^2+ s^2 [d(y_{mk-1},y_{nk})]^2 +2s^2 d(y_{mk},y_{mk-1})\, d(y_{mk-1},y_{nk}) \quad 3.9$$

$$[d(y_{mk-1}, y_{nk})]^2 \leq [sd(y_{mk-1},y_{mk})+ sd(y_{mk},y_{nk})]^2$$

$$= s^2 [d(y_{mk-1},y_{mk})]^2+ s^2 [d(y_{mk},y_{nk})]^2 +2s^2 d(y_{mk-1},y_{mk})\, d(y_{mk},y_{nk}) \quad 3.10$$

In view 3.8, we have

$$\varepsilon^2/s^2 \leq \limsup_{k\to\infty} [d(y_{mk}, y_{nk-1})]^2 \leq \varepsilon^2$$

Using 3.9 and 3.10, we obtain

$$\varepsilon^2/s^2 \leq \limsup_{k\to\infty} [d(y_{mk-1}, y_{nk})]^2 \leq s^4 \varepsilon^2$$

Similarly, we deduce that

$$[d(y_{mk-1}, y_{nk-1})]^2 \le [sd(y_{mk-1},y_{mk}) + sd(y_{mk},y_{nk-1})]^2$$

$$= s^2 [d(y_{mk-1},y_{mk})]^2 + s^2 [d(y_{mk},y_{nk-1})]^2 + 2s^2 d(y_{mk-1},y_{mk}) d(y_{mk},y_{nk-1})$$

$$[d(y_{mk}, y_{nk})]^2 \le [sd(y_{mk},y_{mk-1}) + sd(y_{mk-1},y_{nk})]^2$$

$$= s^2 [d(y_{mk},y_{mk-1})]^2 + s^2 [d(y_{mk-1},y_{nk})]^2 + 2s^2 d(y_{mk},y_{mk-1}) d(y_{mk-1},y_{nk})$$

$$\le \quad s^2 [d(y_{mk},y_{mk-1})]^2 + s^2[s\, d(y_{mk-1},y_{nk-1}) + s\, d(y_{nk-1},y_{nk})]^2 +$$

$$2s^2 d(y_{mk},y_{mk-1})[s\, d(y_{mk-1},y_{nk-1}) + s\, d(y_{nk-1},y_{nk})] \qquad 3.11$$

It follows that

$\varepsilon^2/s^4 \le \limsup_{k\to\infty} [d(y_{mk-1}, y_{nk-1})]^2 \le s^2 \varepsilon^2$

Through the definition of M(x,y), we have

$M(x_{mk},x_{nk}) = \max \{ [d(y_{nk},y_{nk-1})]^2, [d(y_{mk},y_{nk-1})]^2, [d(y_{mk-1},y_{nk-1})]^2, [d(y_{mk},y_{nk-1})]^2[1+[d(y_{mk-1},y_{nk-1})]^2]/1+[d(y_{mk},y_{nk-1})]^2, [d(y_{mk-1},y_{nk-1})]^2[1+[d(y_{mk-1},y_{nk-1})]^2]/1+[d(y_{mk},y_{mk-1})]^2\}$ it is show that

$M(x_{mk},x_{nk}) = \max\{ 0, \varepsilon^2/s^2 ,\varepsilon^2/s^4, \varepsilon^2/s^2, \varepsilon^2/s^2(1+ \varepsilon^2/s^4)\} = \varepsilon^2/s^2$

$\Psi([d(y_{mk},y_{nk})]^2) \le \Psi(s^2[d(y_{mk},y_{nk})]^2) \le \psi(M(x_{mk},x_{nk})) - \varphi(M(x_{mk},x_{nk}))$

$\psi(s^2\varepsilon^2) \le \psi(s^2\varepsilon^2) - \varphi(s^2\varepsilon^2)$ which is contradiction.

It follows that $\{y_n\}$ is a Cauchy sequence in X and $d(y_m,y_n) = 0$.Since X is complete b- metric – like space, there exists u ε X such that

$$\lim_{n\to\infty} d(y_n,u) = \lim_{n\to\infty} d(fx_n,u) = \lim_{n\to\infty} d(gx_{n+1},u) = \lim_{n,m\to\infty} d(y_n,y_m) = d(u,u) = 0 \qquad 3.12$$

Further, we have u ε g(X) since g(X) is ciosed. It follows that one can choose a z ε X such that u = gz, and one can write 3.12 as

$$\lim_{n\to\infty} d(y_n,gz) = \lim_{n\to\infty} d(fx_n,gz) = \lim_{n\to\infty} d(gx_{n+1},gz) = 0.$$

If fz ≠ gz, taking $x = x_{nk}$ and y = z in contractive condition 3.1, we get

$$\Psi(s^2[d(y_{nk},fz)]^2) \quad \le \psi(M(x_{nk},fz)) - \varphi(M(x_{nk},fz)) \qquad 3.13$$

Where

$$M(x_{nk},z) = \max\{ [d(fz,gz)]^2, [d(fx_{nk},gz)]^2, [d(gx_{nk},gz)]^2, \frac{[d(fxnk,gz)]2[1 + [d(gxnk,gz)]2]}{[1 + [d(fxnk,gz)]2]},$$

$$\frac{[d(gxnk,gz)]2[1 + [d(gxnk,gz)]2]}{[1 + [d(fxnk,gz)]2]} \} \qquad 3.14$$

And we obtain $\limsup_{k\to\infty} M(x_n,z)$

$$= \max \{ [d(fz,gz)]^2,0,0,0,0\} = [d(fz,gz)]^2$$

Taking the upper limit as k→∞ in 3.14

$\Psi([d(fz,gz)]^2) \leq \Psi(s^2. 1/s^2 [d(fz,gz)]^2) \leq \Psi(s^2[\limsup_{k\to\infty} d(fx_n,fz)]^2) \leq \psi(\limsup_{k\to\infty} M(x_n,z))$

$- \varphi(\limsup_{k\to\infty} M(x_n,z))$

$= \psi([d(fz,gz)]^2) - \varphi([d(fz,gz)]^2)$, which implies that $\varphi([d(fz,gz)]^2) = 0$. It follows that

$d(fz,gz) = 0$ this implies that fz = gz. Therefore u = fz =gz is a point of coincidence foe f and g. We also conclude that the point of coincidence is unique. Assume on the contrary that there exists z, $z^* \varepsilon$ C(f,g) and $z \neq z^*$, appling 3.1 with x= z and y = z^*, we obtaind that $\Psi([d(fz,fz^*)]^2) = \Psi(s^2[d(fz,fz^*)]^2) \leq \psi(M(z,z^*)) - \varphi(M(z,z^*)) = \psi([d(fz,fz^*)]^2) - \varphi([d(fz,fz^*)]^2)$.

Hence fz = fz^*. That is the point of coincidence is unique. Considering the weak compatibility of f and g, it can be shown that z is a unique common fixed point.

Example

Let X = [0,1] be endowed with the b- metric – like $d(x,y) = (x+y)^2$ for all x,y εx and s = 2. Define mapping f,g : X→X by fx = x/64 and gx= x/2. Control function ψ,φ : [0,∞)→[0,∞) are defined as ψ(t) = 5t/4, φ(t) = 48545t/87846 for all t ε [0,∞). It is clear that f(X) c g(X) is closed. For all x,y ε X , all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of f and g.

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